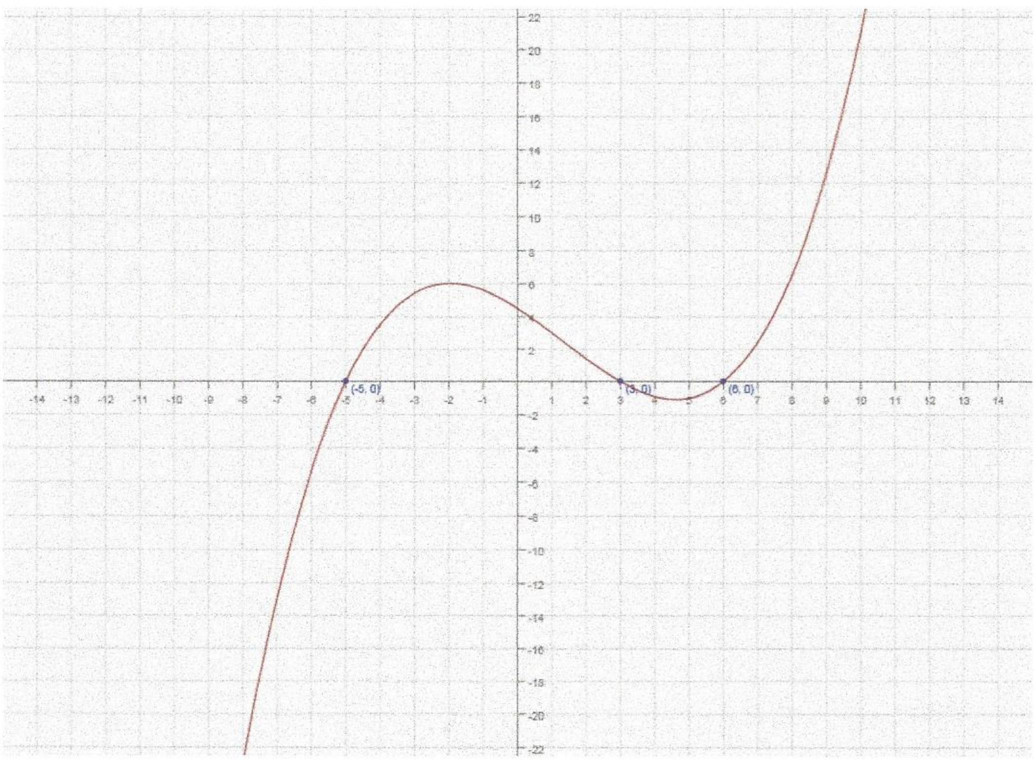
Graphing Methods in Finding the Complex Roots of Cubic Equations

A cubic equation is $ax^3 + bx^2 + cx + d = 0,\ (a \neq 0)$. To solve the equation is the same to find zeros of the corresponding polynomial $f(x) = ax^3 + bx^2 + cx + d$. It is well-known that the cubic equation has at least one real root (see [1]). So there are two cases:

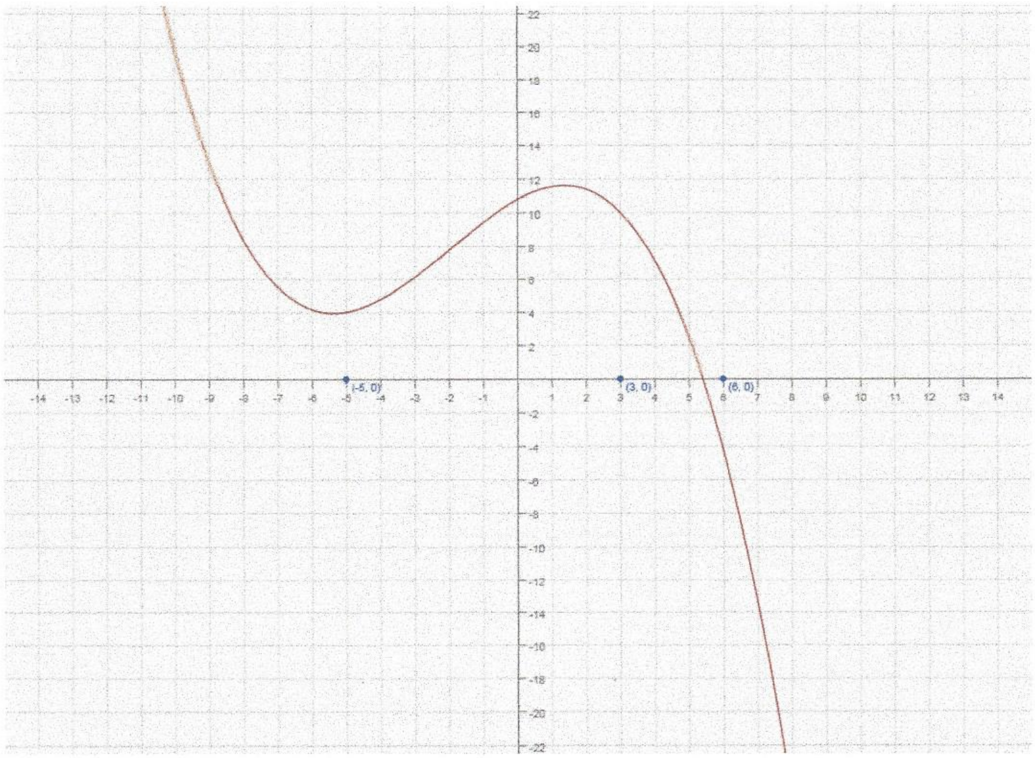
Case 1. There are three (3) real roots (some may be repeated).
Case 2. There is one (1) real root and two conjugate complex roots.

In the first case, the graph of the corresponding polynomial $f(x) = ax^3 + bx^2 + cx + d$ will be something like this:



So the three real zeros can be easily visualized in the graph as the three x-intercepts. And there is a standard procedure that we can use the TI-84 graphing calculator to find the zeros.

On the other hand, in the second case, the graph of the corresponding polynomial $f(x) = ax^3 + bx^2 + cx + d$ will only have one x-intercept. So far, there is no known ways to visualize the two conjugate complex zeros in the graph. (See the figure below.)



Similar situation occurs in quadratic equations. Quadratic equations and quadratic functions are two basic must-have topics in high school algebra and in freshmen college algebra. For a quadratic equation

$$ax^2 + bx + c = 0, \ (a \neq 0),$$

its solutions have the following three cases.

Case 1. The discriminant $D = b^2 - 4ac > 0$, there are two real solutions

$$x_1 = -\frac{b}{2a} + \frac{\sqrt{b^2 - 4ac}}{2a} \text{ and } x_1 = -\frac{b}{2a} - \frac{\sqrt{b^2 - 4ac}}{2a}.$$

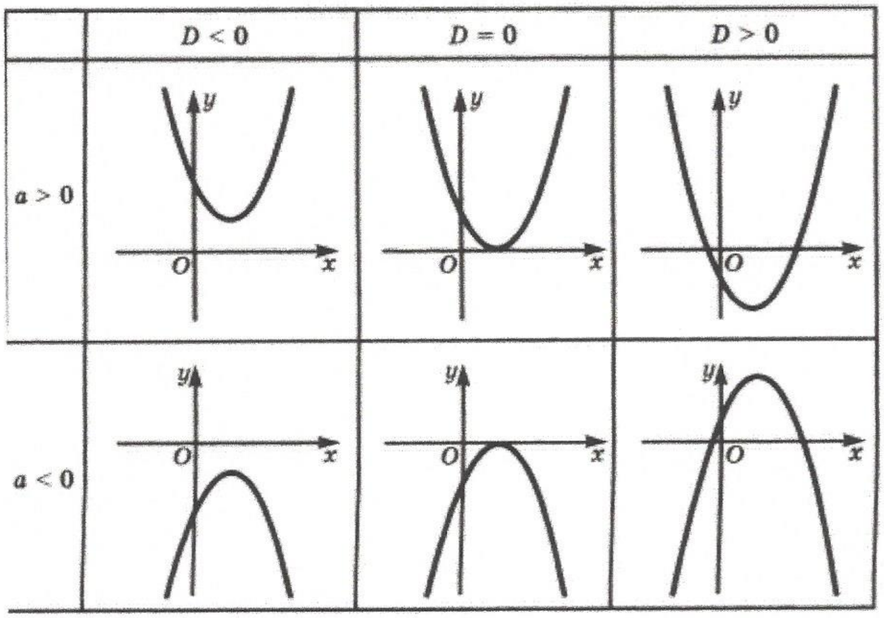
Case 2. The discriminant $D = b^2 - 4ac = 0$, there is one repeated real solutions and $x = -\frac{b}{2a}$.

Case 3. The discriminant $D = b^2 - 4ac < 0$, there is no real solutions, and there are two conjugate complex solutions.

$x_1 = -\frac{b}{2a} + \frac{\sqrt{|b^2 - 4ac|}}{2a} i$ and $x_1 = -\frac{b}{2a} - \frac{\sqrt{|b^2 - 4ac|}}{2a} i$, where $i = \sqrt{-1}$ is the imaginary unit.

There is a nice geometric explanation for all these cases. Especially, the first two cases, the solutions can be found by finding the x-intercepts of the corresponding quadratic functions

$$f(x) = ax^2 + bx + c, \ (a \neq 0)$$



The graphs help students to see where the real solutions are.

The last case when the discriminant $D = b^2 - 4ac < 0$, the graph just shows there is no x-intercept.

So the natural questions are: in the cubic equation or quadratic equation, where are the two conjugate complex roots? Or, can we find the two conjugate complex solutions by identifying some points, just like to identify the x-intercepts for the real roots.

Recently, the research done by Feng-Carthon-Brown [2], found the geometric ways to visualize the two conjugate complex roots for a quadratic equation. In this research proposal, we explore if there is some graphical and intuitive way to find the complex roots of cubic equations, that is, the complex zeros of polynomials of degree 3. In general, there is not any easy way to find complex roots of cubic equations, especially when the roots are not real numbers. The algebraic way to find complex roots of cubic equations is too hard to visualize. The graphical way to find complex roots of cubic equations have not been seen in literature. Our research project is to find such a graphical way to solve cubic equations such that the topic of solving cubic equations can be taught in middle schools and high schools. And also, we are to find such a graphical way that we can use technology (e.g. TI-84 graphing calculator) to solve cubic equations.

Our research methods include, algebra, geometry and technology (graphing calculators).

As there is no known graphical method in finding the complex roots of cubic equations, any find out will be original and has a high potential to be published in refereed research journals such as *Mathematics Magazine*, *The College Mathematics Journal.* The significance and impact of the research projects are:

1. Increase our knowledge in finding complex roots of cubic equations.
2. Enable us to use graphing calculator to find the complex roots of cubic equations.
3. Make the process of finding complex roots of cubic equations geometrically visible.
4. Enable us to teach the topic in college algebra classes, in a high school or a middle school classroom.

References

[1] Ron Irving, *Beyond the Quadratic Formula*, MAA Press, 2013.

[2] L. Feng, J. Carthon and C. Brown, *Visualize the Two Conjugate Complex Roots for Quadratic Equations,* preprint, 2016.